UNIT-2

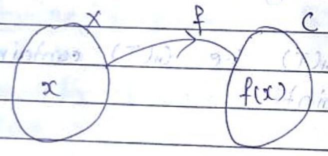
* **linear function :-** let X be a vector space over a field C (real & complex) a mapping $f: X \to C$ is said to be linear function if $x_1, x_2 \in X$ and $\alpha_1, \alpha_2 \in R$

$$f(\alpha_1 x_1 + \alpha_2 x_2) = \alpha_1 f(x_1) + \alpha_2 f(x_2)$$

$$\forall\, x_1, x_2 \in X \text{ and } \alpha_1, \alpha_2 \in R$$

$$\alpha_1 x_1 + \alpha_2 x_2 \in X$$

f is said to be real or complex linear functional according as f is real or complex valued function.



* **Sub-linear Transformation :-** let X and Y be two nls and let a transformation T from X to Y. Then $T: X \to Y$ is said to be sub-linear transformation if

$T(x+y) \le T(x) + T(y) \quad \forall\, x, y \in X$

and $T(\alpha x) = \alpha T x$, where α positive real no.

* **Functional :-** let X be a nls over a field K, where K is real field or complex field. Then the functional is a transformation $T: X \to K$, if K is real then it is called real functional.

We denote X^* for the set of all bounded real functional.

D. 10, 15, 13

* **Hahn-Banach Theorem for real linear space:**

Statement :- let E be a real linear space and let M be a linear subspace ~~functional~~ of E. suppose p is a sub-linear functional define on E and p be linear functional defined on M such that,

$$f(x) \le p(x) \quad \forall\, x \in M.$$

Then there is a linear functional g defined on E such that g is an extension of f (i.e. $g(x) = f(x)\ \forall\, x \in M$) and

$$g(x) \le p(x) \quad \forall\, x \in E$$

Proof :- let F denote the set of all real functions h such that h is linear domain of which h is linear subspace of E, h is an extension of f and

$$h(x) \le p(x) \quad \forall\, x \in \text{dom}\, h$$

since, $f \in F$ and $F \ne \phi$

all write $f \subset h$ to denote that h is an extension of f i.e. $\text{dom} f \subset \text{dom}\, h$ and

$$f(x) = h(x) \quad \forall\, x \in \text{dom} f.$$

Note that F is a partially ordered set w.r.t. the partial order $\leq$.

let c be any chain in F and let,
$h = \cup c$.

Then, $h \in F$
Therefore from zorn's lemma it follows that F has a maximum element g.
we complete the proof by showing that
$\mathrm{dom}\, g = E$.

suppose if possible that this is false
i.e. let
$\mathrm{dom}\, g \leq E$;

let y be any element in $E \cap G$ and then define,
$H = \{x + \alpha y : x \in G, \alpha \in R\}$
clearly H is a linear subspace of E and G be a proper subspace of H. i.e.

$G \subseteq H$.
let c be a fixed but arbitrary real number
define h on H by:-

$h(x+\alpha y) = g(x) + \alpha c$
Now if,
$x_1 + \alpha_1 y = x_2 + \alpha_2 y$

where $x_1, x_2 \in G$ and $\alpha_1, \alpha_2 \in R$ Then,

$(\alpha_1 - \alpha_2) y = x_2 - x_1 \in G$
$\Rightarrow \alpha_1 = \alpha_2$ and $x_1 = x_2$

Hence h is well-defined, and clearly h is linear and $g \subseteq h$.
Now we claim that c can be selected so that,
$h(x) \leq p(x) \quad \forall x \in H$.
Then we have $h \in F$ which contradict the maximality of g and complete the proof of theorem.

Varification:- Therefore we need only to establish that c can be so selected. Justify our claim. Thus our requirement is that,
$g(x) + \alpha c = h(x+\alpha y) \leq p(x+\alpha y)$
$\forall x \in G, \alpha \in R$

since g is linear and p is sublinear. Therefore, it equivalent to requiring that,
$g\left(\frac{x}{\alpha}\right) + c \leq p\left(\frac{x}{\alpha} + y\right) \quad \forall x \in G, \alpha > 0.$
and, $g\left(\frac{x}{\alpha}\right) + c \geq -p\left(\frac{-x}{\alpha} - y\right) \quad \forall x \in G, \alpha < 0.$

Therefore, it is sufficient to have,

$$g(u) - p(u-y) \le c \le -g(v) + p(v+y) \quad \forall\, u, v \in G.$$

But we do have,

$$g(u) + g(v) = g(u+v) \le p(u+v)$$
$$\le p(u-y) + p(v+y)$$
$$\forall\, u, v \in G$$

write,

$$a = \sup\{ g(u) - p(u-y) : u \in G \}$$

and

$$b = \inf\{ -g(v) + p(v+y) : v \in G \}$$

so it clear that,

$$a \le b.$$

Now, any real number c such that, $a \le c \le b$ satisfy our requirement. This complete the proof of theorem.

Q.12 Hahn Banach Theorem for a real normed linear space:

Statement:- let E be a real normed linear space and let M be a linear subspace of E. If $f \in M^*$ then there exist $g \in E^*$ such that $f(x) = g(x)$ $\forall x \in M$ and $\|g\| = \|f\|$.

Proof:- E^* set of all bounded linear functionals defined on E. $f : E \to R$

M^* - set of all bounded linear functional defined on M.

define p on E by

$$p(x) = \|f\|\,\|x\| \quad \forall x \in E \quad \text{——} (1)$$

(i) P is well defined :- for $x_1, x_2 \in E$

$$x_1 = x_2$$
$$\Rightarrow \|x_1\| = \|x_2\|$$
$$\Rightarrow \|f\|\,\|x_1\| = \|f\|\,\|x_2\|$$
$$\Rightarrow p(x_1) = p(x_2)$$

$\Rightarrow$ p is well defined.

(ii) P is sublinear functional :- let $x, y \in E$, $x+y \in E$.

$$p(x+y) = \|f\|\,\|x+y\|$$
$$\le \|f\|(\|x\| + \|y\|) \quad \{\because \text{Triangle inequality}$$
$$= \|f\|\,\|x\| + \|f\|\,\|y\|$$
$$p(x+y) = p(x) + p(y)$$

also for $\alpha \ge 0$ and $x \in E$

$$p(\alpha x) = \|f\|\,\|\alpha x\|$$
$$= |\alpha|\,\|f\|\,\|x\|$$
$$= |\alpha|\, p(x)$$
$$p(\alpha x) = \alpha\, p(x)$$

(iii) $f(x) \le |f(x)|$

$$\le \|f\|\,\|x\| = p(x) \quad \{\because \text{by eqn } (1)\}$$
$$\forall x \in M.$$

This follows that by Hahn banach extension theorem for real linear spaces $\exists$ a linear functional g on E such that $f \subset g$ and.

$$g(x) \le p(x) \quad \forall x \in E$$

Now, we have to prove that g is bounded and

$$\|g\| = \|f\|$$

since $g(x) \le p(x)$ for all $x \in M$
and $p(x) = \|f\| \|x\|$

$$\therefore -g(x) = g(-x)$$
$$\le p(-x)$$
$$= \|f\| \|-x\|$$
$$= \|f\| \|x\|$$
$$= p(x)$$

$\Rightarrow |g(x)| \le p(x)$
$\Rightarrow |g(x)| \le \|f\| \|x\|$

$$\|g\| = \sup\left\{ \frac{|g(x)|}{\|x\|} : x \in E, \|x\| \ne 0 \right\} \le \|f\|$$

$\Rightarrow \|g\| \le \|f\|$ —— ②

again,

$$\|g\| = \sup\{ |g(x)| : x \in E, \|x\| = 1 \}$$
$$\ge \sup\{ |g(x)| : x \in M, \|x\| = 1 \}$$
$$= \sup\{ |f(x)| : x \in M, \|x\| = 1 \}$$
$$\{\because g(x) = f(x), \forall x \in M\}$$

$\|g\| \ge \|f\|$ —— ③

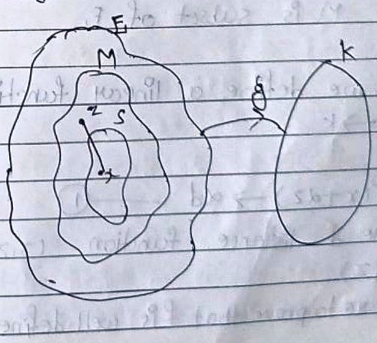
combining eqn ② and ③ we get,

$$\boxed{\|g\| = \|f\|}$$

Therefore $\|g\| = \|f\|$ and this complete the proof.

16.10.15
* 165 H.K. **Theorem:-** let E be a nls over field K and let S be a linear subset of E. suppose that $z \in E$ and distance between $(z, S) = d > 0$. Then $\exists$ $g \in E^*$ such that $g(S) = 0$, $g(z) = d$, $\|g\| = 1$.

Proof:-



E^* is set of all bounded linear functional defined on E.
let we define,

$M = \{x + \alpha z : x \in S, \alpha \in K\}$ where $z \in E \cap S^c$

Then M is subset of E, for this $x_1, x_2 \in S$ and $\alpha_1, \alpha_2 \in K$ Then, $x_1 + \alpha_1 z$ and $x_2 + \alpha_2 z \in M$.

Now, $d_1, d_2 \in K$ we have,

$d_1(x_1 + \alpha_1 z) + d_2(x_2 + \alpha_2 z) = (d_1 x_1 + d_2 x_2) + (d_1\alpha_1 + d_2\alpha_2) z$

Since $d_1 x_1 + d_2 x_2 \in S$ (because S is sublinear of E) and $(d_1\alpha_1 + \alpha_2 d_2) \in K$.

$\therefore d_1(x_1 + \alpha_1 z) + d_2(x_2 + \alpha_2 z) \in M$
Hence M is subset of E.

Now, we define a linear function $f: M \to K$

$f(x + \alpha z) \to \alpha d$ ——— ①
where d distance function (x, z) or (S, z)
we are to prove that f is well defined function

① f is well defined :- let $x_1 + \alpha_1 z$ and $x_2 + \alpha_2 z \in M$ then,

$x_1 + \alpha_1 z = x_2 + \alpha_2 z$
$\Rightarrow x_1 - x_2 = (\alpha_2 - \alpha_1) z$.
since $x_1, x_2 \in S$ but $z \notin S$ therefore $\alpha_2 - \alpha_1 = 0$
$\Rightarrow x_1 - x_2 = 0$
$\Rightarrow x_1 = x_2$ and $\alpha_1 = \alpha_2$.

Hence, $\alpha_1 d = \alpha_2 d$
$\Rightarrow f(x_1 + \alpha_1 z) = f(x_2 + \alpha_2 z)$ {by eqn ①}

② f is linear :- let $x_1, x_2 \in S$ and $\alpha_1, \alpha_2 \in K$
Then $x_1 + \alpha_1 z$ and $x_2 + \alpha_2 z \in M$ such that $f(x_1 + \alpha_1 z) = \alpha_1 d$ and $f(x_2 + \alpha_2 z) = \alpha_2 d$.

Now, we defined $d_1, d_2 \in K$

$f[d_1(x_1 + \alpha_1 z) + d_2(x_2 + \alpha_2 z)]$
$= f[\underbrace{(d_1 x_1 + d_2 x_2)}_{x} + \underbrace{(d_1\alpha_1 + d_2\alpha_2)}_{\alpha} \underbrace{z}_{z}]$
$= (d_1\alpha_1 + d_2\alpha_2) d$ {$\because$ by ①}
$= d_1\alpha_1 d + d_2\alpha_2 d$
$= d_1 f(x_1 + \alpha_1 z) + d_2 f(x_2 + \alpha_2 z)$ {by ①}
$\therefore$ f is linear on M.
Also we have,
$f(x + \alpha z) = \alpha d \quad \forall x \in S$
but $\alpha = 0$ we get,
$f(x) = 0 \Rightarrow f(S) = 0$

again putting $x=0$ and $\alpha=1$ we get
$f(z)=d$

further.

$$\|f\| = \sup\left\{\frac{|f(x+\alpha z)|}{\|x+\alpha z\|} : x+\alpha z \in M, \|x+\alpha z\| \neq 0\right\}$$

$$= \sup\left\{\frac{|\alpha d|}{\|x+\alpha z\|} : x+\alpha z \in M, \|x+\alpha z\| \neq 0\right\}$$

$$= d \sup\left\{\frac{\alpha}{\|x+\alpha z\|} : x+\alpha z \in M, \alpha \neq 0\right\}$$

$$= d \sup\left\{\frac{\alpha}{\alpha\{x/\alpha + z\}} : x+\alpha z \in M, \alpha \neq 0\right\}$$

$$= d \sup\left\{\frac{1}{(\frac{x}{\alpha}+z)} : x+\alpha z \in M, \alpha \neq 0\right\}$$

$$\|f\| = \left\{\frac{d}{\inf\{x/\alpha + z\}} : x/\alpha + z \neq 0\right\}$$

$$= \left\{\frac{d}{\inf(y+z)} : y+z \neq 0, \ x/\alpha = y, \ x/\alpha \in S\right\}$$

$$\|f\| = \frac{d}{d} = 1 \quad \{\because d = \inf(y+z)\}$$

$\therefore f$ is bounded.

Hence $f \in M^*$ Now applying Hahn Banach Theorem we can find $g \in E^*$ such that

$f(y) = g(y) \ \forall y \in M$
and $\|f\| = \|g\|$
since $S \subset M$ therefore
$f(S) = g(S) = 0$
$f(z) = g(z) = d$
$\|f\| = 1 = \|g\|$

* Natural Mapping (Canonical Embedding)

let E be a nls then a mapping $\pi : E \to E^{**}$ such that $\pi(x) = F_x \ \forall x \in E$ and $F_x \in E^{**}$ where $F_x : E^* \to K$ such that
$F_x(f) \to f(x) \quad \forall f \in E^*$
is said to be canonical embedding (natural mapping) from E into E^{**}

F_x is linear and bounded

(i) $F_x(f+g) = (f+g)(x)$ for $f, g \in E^*$
$= f(x) + g(x)$
$F_x(f+g) = F_x(f) + F_x(g)$

(ii) $F_x(\alpha f) = (\alpha f)(x) = \alpha f(x) = \alpha F_x(f)$

again,

$$\|F_x\| = \sup\left\{\frac{|F_x(f)|}{\|f\|} : f \in E^* \text{ and } \|f\| \neq 0\right\}$$

$$= \sup\left\{\frac{|f(x)|}{\|f\|} : f \in E^* \text{ and } \|f\| \neq 0\right\}$$

$\leq \sup\left\{\frac{\|f\|\|x\|}{\|f\|} : f \in E^* \text{ and } \|f\| \neq 0\right\}$

$\|F_x\| \leq \|x\|$

$\therefore F_x$ is linear and bounded.

* Theorem :- let E be a n/s let π be a natural mapping from E onto E^{**}. Then π is norm preserving, bounded linear transformation from E in E^{**} also π is one-one.

Proof :- let π be a mapping $\pi : E \to E^{**}$ such that $\pi(x) = F_x$ $x \in E$

and $F_x : E^* \to K$ such that

$F_x(f) = f(x)$ $\forall f \in E^*$ and $x \in E$

(1) π is linear :- let $x, y \in E$ and $\alpha, \beta \in K$ then $\alpha x + \beta y \in E$ (because E is n/s)

and let $f \in E^*$

Now,

$\pi(\alpha x + \beta y) f = F_{(\alpha x + \beta y)}(f)$

$= f(\alpha x + \beta y)$

$\therefore f$ is bounded linear functional)

$= \alpha F_x(f) + \beta F_y(f)$

$\pi(\alpha x + \beta y) f = \alpha \pi(x) f + \beta \pi(y) f$

$\Rightarrow \pi(\alpha x + \beta y) = \alpha \pi(x) + \beta \pi(y)$

$\therefore \pi$ is linear.

(2) π is well defined :- If $x = y$ then we have to show that

$\pi(x) = \pi(y)$

$\therefore f$ is well defined and linear.

$\because x = y$

$\Rightarrow f(x) = f(y)$

$\Rightarrow F_x(f) = F_y(f)$

$\Rightarrow F_x = F_y$

$\Rightarrow \pi(x) = \pi(y)$

$\therefore \pi$ is well defined.

(3) π is bounded :- since

$\|\pi(x)\| = \|F_x\|$

$\|\pi(x)\| \leq \|x\|$ —— (1)

$\because \|\pi(x)\| = \sup\left\{\frac{\|\pi(x)\|}{\|x\|} ; \|\pi\| \neq 0\right\}$

$\leq \sup\left\{\frac{\|\pi\| \|x\|}{\|x\|} ; \|\pi\| \neq 0\right\}$

$\|\pi(x)\| < 1$

$\therefore \pi$ is bounded.

(4) π is norm preserving :- corollary: let E be a n/s and z be a non-vector in E then $\exists$ a function $g \in E^*$ s.t. $g(z) = \|z\|$ and $\|g\| = 1\}$

let $x \in E$ be any non-zero element then by the corollary $\exists$ $g \in E^*$ s.t. $\|g\| = 1$ and $g(x) = \|x\|$.

Now,

$\|x\| = g(x) \le \sup\{|f(x)| : f \in E^* \text{ and } \|f\| \le 1\}$

$= \sup\{|F_x(f)| : f \in E^* \text{ and } \|f\| \le 1\}$

$\le \|F_x\| \le \|\pi(x)\|$

$\|x\| \le \|\pi(x)\|$ —— (2)

combine eqn (1) and (2) we get,

$\|x\| = \|\pi(x)\|$

$\therefore \pi$ is norm preserving.

(5) π is one-one :- let $x, y \in E$ and suppose that $x \ne y$ then

$\|\pi(x) - \pi(y)\| = \|\pi(x-y)\|$

$= \|x-y\| \ne 0 \quad \{\because \|\pi(x)\| = \|x\|\}$

$\|\pi(x) - \pi(y)\| \ne 0$

$\pi(x) \ne \pi(y)$

$\therefore \pi$ is one-one.

* Reflexive space :- A normed linear space X is said to be reflexive in $\exists$ a natural mapping (embedding) π from X onto X^{**} i.e. $\pi : X \to X^{**}$,

Then $\pi(X) = X^{**}$

exa (1) R^n is a reflexive space

(2) L^p is a reflexive space, if $1 < p < \infty$.

(3) finite dimensional complete normed linear space is reflexive.

* Theorem :- let X be a reflexive nls then

(i) X is Banach and it remains reflexive in any equivalent norm.

(ii) X^{**} is reflexive.

(iii) every closed subset of X is reflexive.

Proof :- (i) since the dual X^* and the second dual X^{**} of the nls X are banach space.

If X is reflexive then X is linearly isometric to X^{**} so that X is Banach space.

$\pi : X \to X^{**}$

$\pi(X) = X^{**}$

$\Rightarrow X \simeq X^{**}$

also in any equivalence norm on X the dual X^* and the second dual X^{**} remains unchanged. so that X remain reflexive

(ii) let X be reflexive then $\exists$ a natural embedding $\pi : X \to X^{**}$ such that π is onto.

we claim X^* is reflexive.

let $\Pi^*: X^* \to X^{***}$ then obviously Π^* is linear bounded and one-one.

our aim is show that Π^* is onto

let $x^{***} \in X^{***}$ defined $x^* \in X^*$ by

$$x^*(x) = x^{***}(\Pi^{-1}(x)) \text{ for } x \in X$$

we claim that $\Pi^*(x^*) = x^{***}$

take in arbitrary $x^{**} \in X^{**}$

since X is reflexive

$$x^{**} = \Pi(x) \text{ for } x \in X$$

Then,

$$x^{***}(x^{**}) = x^{***}(\Pi(x))$$

$$\Rightarrow x^{***}(x^{**}) = x^*(x)$$

$$\Rightarrow \Pi^*(x^*) = x^{***}$$

$\Rightarrow$ Hence X^* is reflexive.

Conversely :- let X^* is reflexive.

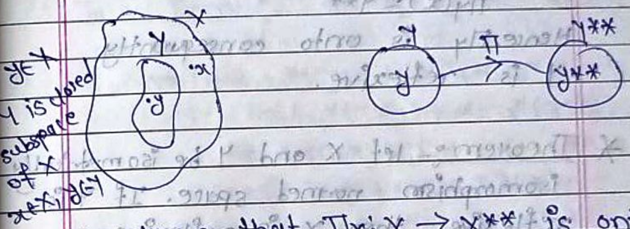
Thus X^{**} is reflexive and $\Pi(X)$ is closed sub-space of X^{**} and Hence, reflexive.

since $\Pi(x) = x$

$\therefore$ X is reflexive.

(iii) let Y be a closed subspace of X and $\Pi_Y : Y \to Y^{**}$

we claim that Π_Y is onto



we know that $\Pi_X : X \to X^{**}$ is onto because X is reflexive.

let $y^{**} \in Y^{**}$ be any arbitrary element

define $x^{**} \in X^{**}$ by

$$x^{**}(x^*) = y^{**}(x^*/y)$$

where x^*/y is restriction of X^* on Y.

since, Π_X is onto (because X is reflexive)

$\exists$ an $x \in X$ s.t. $\Pi_X(x) = x^{**}$.

Now, we again claim $x \in Y$ suppose $x \notin Y$.

then since Y is closed by corollary of Hahn Banach Theorem $\exists$ $x_0^* \in X$ such that but $x_0^*/y = 0$

But,

$$x_0^*(x) = d(x, y) \neq 0$$

Thus we have,

$$0 \neq d(x, y) = x_0^*(x)$$
$$= x^{**}(x^*) = y^{**}(x^*/y)$$

$= y^{**}(0) = 0$

which is contradiction.

Hence $x \in Y$ and we have

$T/y(x) = y^{**}$

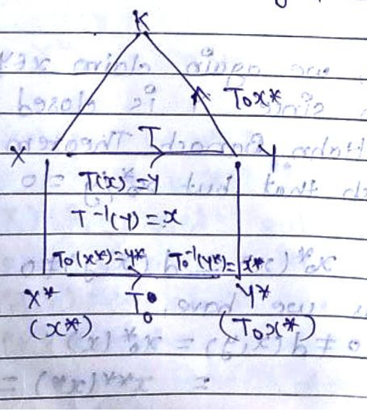
Hence, T/y is onto consequently Y is reflexive.

✱ Theorem :- let X and Y be isometrically isomorphism normed space. If Y is reflexive then X is reflexive.

Proof :- let $T : X \to Y$ be an isometric isomorphism of X onto Y. consider the map $T_0 : X^* \to Y^*$ defined by

$(T_0 x^*)(y) = x^*(T^{-1}y)$ —— (i)

where $x^* \in X^*$ and $y \in Y$.



Now, we shall varify that T_0 is norm preserving isomorphism of X^* onto Y^*.

(1) To is linear :- for $x_1^*, x_2^* \in X^*$ and $\alpha_1, \alpha_2 \in K$ we have,

$$T_0(\alpha_1 x_1^* + \alpha_2 x_2^*)(y) = (\alpha_1 x_1^* + \alpha_2 x_2^*)(T^{-1}y) \quad \{\text{by (i)}\}$$
$$= \alpha_1 x_1^*(T^{-1}y) + \alpha_2 x_2^*(T^{-1}y)$$
$$= \alpha_1 (T_0 x_1^*)(y) + \alpha_2 (T_0 x_2^*)(y)$$
$$= (\alpha_1 T_0 x_1^* + \alpha_2 T_0 x_2^*)(y)$$

$T_0(\alpha_1 x_1^* + \alpha_2 x_2^*) = (\alpha_1 T_0 x_1^* + \alpha_2 T_0 x_2^*)$

$\Rightarrow$ T_0 is linear.

(2) To is bounded and norm preserving :-

for this

$$\|T_0 x^*\| = \sup\{|(T_0 x^*)y| : y \in Y \text{ and } \|y\| = 1\}$$
$$= \sup\{|x^*(T^{-1}y)| : y \in Y \text{ and } \|y\| = 1\}$$
$$\leq \sup\{\|x^*\| \|T^{-1}y\| : y \in Y \text{ and } \|y\| = 1\}$$
$$= \|x^*\| \sup\{\|T^{-1}y\| : y \in Y \text{ and } \|y\| = 1\}$$
$$= \|x^*\| \quad \{\because \|T^{-1}y\| = 1,\ T^{-1} \text{ is bounded and preserving norm}\}$$

$\Rightarrow \|T_0 x^*\| \leq \|x^*\|$ —— (ii)

sine x^* is bounded

$\therefore$ T_0 is bounded.

Now, for a fixed $y_0 \in Y$ and $\|y_0\| = 1$

we have,

$$\|T_0 x^*\| \geq |T_0 x^* y_0| = |x^*(T^{-1}y_0)| \quad \{\because \text{by (i)}\}$$
$$= |x^*(x_0)| \quad \{\because T^{-1} \text{ is onto } \exists\, x_0 \in X \text{ s.t. } x_0 = T^{-1}y_0\}$$

$\|T_0 x^*\| \geq \|x^*\|$ —— (iii)

Hence, combining eqn (ii) and (iii) we get,

$$\|T_0 x^*\| = \|x^*\|$$

(3) **To is one-one and onto :-** let $x \in X$ since T is onto then $\exists\ y \in Y$ s.t. $T^{-1}(y) = x$

let x_1^* and $x_2^* \in X^*$ s.t.

$$x_1^* \neq x_2^* \Rightarrow x_1^*(x) \neq x_2^*(x)$$
$$\Rightarrow x_1^*(T^{-1}y) \neq x_2^*(T^{-1}y)$$
$$\Rightarrow (T_0 x_1^*)(y) \neq (T_0 x_2^*)(y)$$
$$\Rightarrow T_0 x_1^* \neq T_0 x_2^*$$

$\Rightarrow T$ is one-one.

again, for $y^* \in Y^*$ $\exists$ $y \in Y$ and $x^* \in X$ s.t.

$$y^*(y) = x^*(T^{-1}y) = (T_0 x^*)(y)$$

Hence, for any $y^* \in Y^*$ $\exists$ $x^* \in X^*$ s.t.

$\Rightarrow T$ is onto.

Next, we define $T_1 : X^{**} \to Y^{**}$ by

$$(T_1 x^{**})y^* = x^{**}(T_0^{-1} y^*) \quad \text{—— (iv)}$$

where $x^{**} \in X^{**}$ and $y^* \in Y^*$.

likewise T_0 it can be varified that T_1 is isometric isomorphism on X^{**} onto Y^{**}

Next Part :- let $\pi_X : X \to X^{**}$ and $\pi_Y : Y \to Y^{**}$ be natural embedding

If Y is reflexive then π_Y is onto i.e.

$$\pi_Y(Y) = Y^{**}$$

we claim that π_X is onto for this let $x_0^{**} \in X^{**}$ be an arbitrary the fixed element.

set $x_0 = T^{-1}\pi_Y^{-1} T_1 x_0^{**}$ clearly $x_0 \in X$ further, $x^* \in X^*$ we have,

$$\pi^*(x_0)x^* = x^*(T^{-1}\pi_Y^{-1} T_1 x_0^{**})$$
$$= (T_0 x^*)(\pi_Y^{-1}(T_1 x_0^{**})) \quad \{\text{by (1)}\}$$
$$= (T_1 x_0^{**})(T_0 x^*)$$
$$\{\because \pi_Y \text{ is isometric isomorphism}\}$$
$$= x_0^{**}(T_0^{-1}(T_0 x^*)) \quad \{\because \text{by (1)}\}$$
$$= x_0^{**}(x^*)$$

Thus,

$$\pi_X(x_0)x^* = x^*(x_0) = x_0^{**}(x^*)$$
$$\Rightarrow \pi_X(x_0) = x_0^{**}$$

since x_0^{**} was arbitrary Therefore π_X is onto i.e.

$$\pi_X(X) = X^{**}$$

$\Rightarrow X$ is reflexive.

* **Transpose of F :-** let X and Y be normed spaces and $F \in B(X, Y)$ (set of all bounded linear transformation from X onto Y)

define a map $F^*: Y^* \to X^*$ as follows for $y^* \in Y^*$ and $x \in X$.

let $F^*(y^*)x = y^*(F(x))$ Then, clearly F^* is linear x is continuous (bounded). since

$$\|F^* y^*\| \leq \|y^*\| \, \|F^*\| \quad \forall \; y^* \in Y^*$$

Then the map F^* is called transpose of F.

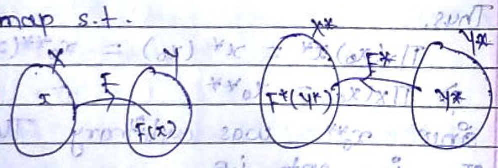
* <u>Theorem</u>:- let X and Y be normed linear space and $F \in B(X, Y)$ then.

(a) $Z(F)$ = zero space of F.

$$= \{x \in X : x^*(x) = 0 \;\forall\; x^* \in R(F^*)\}$$

where $R(F^*)$ range space of F^*.

<u>Proof</u>:- let $F: X \to Y$ and $F^*: Y^* \to X^*$ be linear map s.t.



$F^*(y^*)x = y^*(F(x))$ for $x \in X$ and $y^* \in Y^*$

let $x \in X$ then $F(x) = 0 \Leftrightarrow F^*(y^*)(x)$
$= y^*(F(x)) = y^*(0) = 0$
$\forall\; y^* \in Y^*$ linear for all 0
$y^* \in Y^*$ $\exists\; x^* \in X^*$ such that,

$F^*(y^*) = x^*$

$\therefore F(x) = 0 \Leftrightarrow x^*(x) = 0 \;\forall\; x \in R(F^*)$

Hence, $Z(F) = \{x \in X : x^*(x) = 0 \;\forall\; x^* \in R(F^*)\}$

(b) $Z(F^*) = \{y \in Y^* : y^*(y) = 0 \;\forall\; y \in R(F)\}$
in particular F^* is one-one if and only if $R(F)$ is dense in Y.

<u>Proof</u>:- let $y^* \in Y^*$ be arbitrary $F^*(y^*) = 0$
$\Leftrightarrow y^*(F(x)) = F^*(y^*)x = 0$
for every $x \in X$
since $\forall\; x \in X \;\exists\; y \in R(x)$ s.t.
$$y = F(x)$$

$\because\; y^*(F(x)) = 0 \Leftrightarrow y^*(y) = 0 \;\forall\; y \in R(F)$

$\therefore Z(F^*) = \{y^* \in Y : y^*(y) = 0 \;\forall\; y \in R(F)\}$

Now, F^* is one to one item

$$Z(F^*) = \{0\} \Leftrightarrow y^* = 0.$$

whenever $y^*(y) = 0$ for every $y \in R(F)$.
we know that this happens if and only if the closure of $R(F)$ in Y i.e.

$$\overline{R(F)} = Y.$$

Hence, $R(F)$ is dense in Y.

(c) $R(F^*) \subset \{y \in Y : y^*(y) = 0 \ \forall\, y^* \in Z(F^*)\}$ where equality holds if $R(F)$ is closed in Y.

Proof:- let $y \in R(F)$ and $y = F(x)$ for some $x \in X$ if $y^* \in Z(F^*)$ then

$y^*(y) = y^*(F(x)) = F^*(y^*)(x) = 0$

since, $y \in R(F)$ be arbitrary.

$\therefore R(F) \subset \{y \in Y^* : y^*(y) = 0 \ \forall\, y^* \in Z(F^*)\}$

Now, we shall prove that equality hold in this inclusion if $R(F)$ is closed in Y.

First Part:- suppose equality holds then we can write

$R(F) = \{\cap Z(y^*) : y^* \in Z(F^*)\}$

$\because$ each $Z(y^*)$ is a closed subspace of Y and we know that arbitrary intersection of closed sets in Y is closed hence $R(F)$ is closed in Y.

Conversely:- Assume that $R(F)$ is closed in Y. let $y_0 \notin R(F)$ then there is some $y^* \in Y^*$ s.t.

$y^*(y) = 0$

But by define as above $y^*(y) \neq 0$ for every $y \in R(F)$ in particular

$F^*(y^*)(x) = y^*(F(x)) = 0$

$\forall\, x \in X$ that $y^* \in Z(F^*)$ this shows that

$y_0 \in \{y \in Y : y^*(y) = 0 \ \forall\, y^* \in Z(F^*)\}$

Hence, $R(F) = \{y \in Y : y^*(y) = 0 \ \forall\, y^* \in Z(F^*)\}$

(d) $R(F^*) \subset \{x^* \in X^* : x^*(x) = 0 \ \forall\, x \in Z(F)\}$ where equality holds if X and Y are banach space and $R(F)$ is closed in Y.

Proof:- let $x^* \in R(F^*)$ be arbitrary and $x^* = F^*(y^*)$ for some $y^* \in Y^*$

If $x \in Z(F)$ then $x^*(x) = F^*(y^*)(x)$

$= y^*(F(x))$

$= y^*(0)$

$\Rightarrow x^*(x) = 0$

since, $x^* \in R(F)$ was arbitrary.

Hence, $R(F^*) \subset \{x^* \in X^* : x^*(x) = 0 \ \forall\, x \in Z(F)\}$

Now we are to show that equality holds if X and Y are banach spaces and $R(F)$ is closed in Y.

suppose that X and Y are banach space and $R(F)$ is closed in Y.

let $x^* \in X^*$ be s.t. $x^*(x) = 0$ whenever $F(x) = 0$ then we can find $y^* \in Y^*$ s.t.

$$F^*(y^*) = x^*$$

i.e.

$$y^*(F(x)) = x^*(x) \text{ for every } x \in X$$

define $g : R(F) \to K$ (field) by

$$g(y) = x^*(x) \quad \text{if } y = F(x)$$

Then $x^*(x) = 0 \ \forall\, x \in Z(F)$

The arbitrary g is well defined and linear also the map $F : X \to R(F)$ is linear continuous and onto where X is banach space and so it is the closed subspace $R(F)$ of the banach space Y.

Hence by open mapping theorem and its lemma there is some $x \in X$ with $F(x) = y$ and

$$\|x\| \le Y \|y\| \quad \text{—(1)}$$

$$|g(y)| = |x^*(x)| \le \|x^*\| \|x\| \le \|x^*\| Y \|y\| \quad \{\because \text{by (1)}\}$$

$$\|g\| = \sup \left\{ \frac{|g(y)|}{\|y\|} : \|y\| = 0, y \in Y \right\}$$

Hence, g is banach space and therefore continuous linear functional on $R(F)$.

By Hahn Banach extension theorem there is some $y^* \in Y^*$ s.t. $y^*|R(F) = g$, y^* extension of range of F

$$F^*(y^*)(x) = y^*(F(x))$$
$$= g(F(x))$$
$$= x^*(x) \ \forall\, x \in X$$

$\therefore F^*(y^*) = x^*$ Hence Proved.

13 Closed Range Theorem

Statement: let X and Y be Hilbert space and $T \in B(X, Y)$. Then $R(T)$ is closed if and only if $R(T^*)$ is closed.

Proof: first suppose that $R(T) = Y_0$ is closed

let $T_0 : X \to Y_0$ is defined by,

$$T_0 x = Tx \ \forall\, x \in X.$$

since, T is bounded linear so it is clear that T_0 is a bounded linear operator from X onto Y_0.

Now, we shall show that $R(T_0^*)$ is closed and $R(T_0^*) = R(T^*)$

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(i) To prove that $R(T_0^*)$ is closed :-

Since $T_0 : X \to Y_0$ is a surjective bounded linear operator.

By the result "let X and Y be the Banach spaces and $T \in B(X,Y)$ if T is surjective then $\exists$ $k > 0$. s.t. for every $y \in Y$ $\exists$ $x \in X$ s.t.

$Tx = y$, $\|x\| \le k\|y\|$

It follows that $\exists k > 0$ such that for every $y \in Y_0$ there exist $x \in X$ satisfies

$T_0 x = y$

and $\|x\| \le k\|y\|$ —— (1)

Thus for every $u \in Y_0$ we have,

$|\langle y, u\rangle| = |\langle T_0 x, u\rangle|$

$= |\langle x, T_0^* u\rangle|$ {by def^n of adjoint operator}

$\le \|x\| \, \|T_0^* u\|$

$\le k\|y\| \, \|T_0^* u\|$

$\Rightarrow \dfrac{|\langle y, u\rangle|}{\|y\|} \le k\|T_0^* u\|$

$\Rightarrow \|u\| \le k\|T_0^* u\|$

$\Rightarrow T_0^*$ is bounded below

consequently, by the result "Suppose X is a banach space and $T : X \to Y$ is a bounded linear operator if T is bounded below then $R(T)$ is closed subspace of X.

(ii) To prove that $R(T_0^*) = R(T^*)$

let $x \in X$, $y \in Y_0$

Then $\langle T_0 x, y\rangle = \langle Tx, y\rangle$

$\Rightarrow \langle x, T_0^* y\rangle = \langle x, T^* y\rangle$

$\Rightarrow \langle x, T_0^* y\rangle - \langle x, T^* y\rangle = 0$

$\Rightarrow \langle x, T_0^* y - T^* y\rangle = 0$

$\Rightarrow T_0^* y = T^* y$

since Y_0 is closed therefore by projection theorem.

$Y = R(T) + R(T)^{\perp}$

since we have,

$R(T_0)^{\perp} = N(T^*)$ {Nulity of T^*}

$R(T_0^*) = \{T_0^* y : y \in R(T)\}$

$= \{T^* y : y \in R(T)\}$

$= R(T^*)$

$\Rightarrow \boxed{R(T_0^*) = R(T^*)}$

Conversely :- The converse follows from the 1^{st} part and the fact that

$(T^*)^* = T$

Indeed we have,

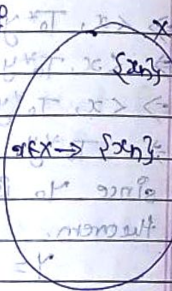
$\Rightarrow R(T)$ is closed

$\Rightarrow R(T^*)$ is closed.

i.e. $R(T^*)$ is closed $\Rightarrow R(T)$ is closed.

* **Compact:-** let X be a normed linear space. then X is said to be compact if every open cover of X has a finite sub cover.

* **Sequentially Compact:-** The normed linear space X is said to be sequentially compact if every sequence in X has a ~~compact~~ converges subsequence converging to an element of X



* **Frechet compact:-** (Bolzano weierstras property)
The normed linear space X is said to be frechet compact if every infinite set in X has a limit point X.

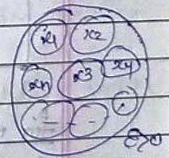
* **Weak converges:-** let X be normed linear space and $\{x_n\}$ a sequence X then $\{x_n\}$ is said to be converges weakly to an element x in X. if for every. $f(x_n) \rightarrow f(x)$ $n \rightarrow \infty$
$x^* \in X^*$, $x^*(x_n) \rightarrow x^*(x)$ as $n \rightarrow \infty$
Then we write $x_n \xrightarrow{\omega} x$

* **Weak* converges:-** let X^* be the dual of X and $\{x_n^*\}$ be a sequence in X^* then the sequence $\{x_n^*\}$ is said to be weak* convergent if $\exists$ $x^* \in X$ s.t.
$x_n^*(x) \rightarrow x^*(x)$ as $n \rightarrow \infty$ $\forall x \in X$.
and we write $x_n^* \xrightarrow{w^*} x^*$ where x^* is weak limit of $\{x_n^*\}$
{ weak and weak* limit are unique.

* **Strong Converges:-** A sequence $\{x_n\}$ in a normed linear space X is said to be strongly converges if $\exists$ $x \in X$ s.t.
$x_n \rightarrow x$ as $n \rightarrow \infty$
or $\|x_n - x\| \rightarrow 0$ as $n \rightarrow \infty$

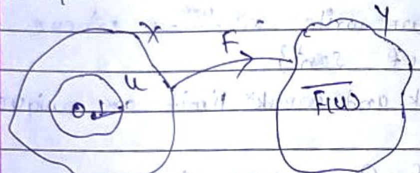
* **Totally bounded:-** let X be n/s then X is said to be totally bounded if for each $\varepsilon > 0$ $\exists$ finite number of elements $x_1, x_2, \dots, x_n$ in X s.t.

$$X = \bigcup_{n=1} B(x_n, \varepsilon)$$



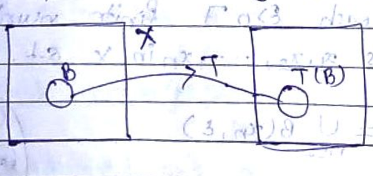
where $B(x_n, \varepsilon)$ is an open ball of radius ε and center at x_n
The finite set $\{x_1, x_2, \dots, x_n\}$ is said to be ε-n/s of X.

* **Compact map:-** let x and y be two normed linear space a map $F: X \to Y$ is said to be compact if $\overline{F(U)}$ is compact in Y where u is open ball $B(0,1)$ in X.



A map is continuous.

* **Compact Operator:-** let X and Y be norm linear space and let T be the mapping from X to Y and a linear transformation, then T is said to be compact operator if $\overline{T(B)}$ is compact for every bounded set $B \subset X$.



for exa:- let X be a norm linear space then $f \in X^*$ is a compact linear transformation on X to C.
In fact if $\{x_n\}$ is any bounded sequence in X then by continuity of $f\{x_n\}$ is a bounded sequence in C. But then $\{f(x_n)\}$ has a convergent subsequence. Thus f is compact linear transformation.

* State and prove Hahn-Banach Theorem for complex normed linear space.

Statement:- "if M is a linear subspace of a complex normed linear space X. and if f is a bounded linear functional on M then f can be extended to a bounded linear functional f_0 defined on the whole space X"

$\|f_0\| = \|f\|$

Proof:- let X be a complex normed linear space and f be a complex valued linear functional on M with $\|f\| = 1$.

put $f(x) = g(x) + ih(x) \quad \forall x \in M$

where g and h are real valued function on M.

To prove that g, h are linearly for $x, y \in M$.

$$\begin{aligned} f(x+y) &= f(x) + f(y) \\ &= g(x) + ih(x) + g(y) + ih(y) \\ &= \{g(x) + g(y)\} + i\{h(x) + h(y)\} \\ &= g(x+y) + ih(x+y) \end{aligned}$$

further if $\alpha \in R$ then,

$f(\alpha x) = \alpha f(x)$
$= \alpha [g(x) + ih(x)]$
$\Rightarrow g(\alpha x) + ih(\alpha x) = \alpha g(x) + i\alpha h(x)$

Hence g and h are linear.

Now, the relations

$f(ix) = if(x)$
$\Rightarrow f(ix) = g(ix) + ih(ix)$
and
$if(x) = ig(x) - h(x)$.

show that,
$h(x) = -g(ix)$

Hence,
$f(x) = g(x) - ig(ix)$

since $\|f\| \leq 1$ we have $\|g\| \leq 1$.
By the Hahn Banach theorem for real n/s $\exists$ an extension g_0 on $M_0 = M + \{x_0\}$ $\forall x_0 \notin M$
with
$\|g_0\| = \|g\| \leq 1$

Define a functional f_0 on M_0 by

$f_0(x) = g_0(x) - ig_0(ix) \quad \forall x \in M_0$.

Now, if α, β are real and $x, y \in M_0$ the we have

$f_0(\alpha x + \beta y) = g_0(\alpha x + \beta y) - ig_0(i(\alpha x + \beta y))$

$= [g_0(\alpha x) - ig_0(i\alpha x)] + [g_0(\beta y) - ig_0(i\beta y)]$

$= \alpha f_0(x) + \beta f_0(y)$

and
$f_0(ix) = g_0(ix) - ig_0(-ix)$
$= g_0(ix) + ig_0(x)$
$= i[-ig_0(ix) + g_0(x)]$
$= i[g_0(x) - ig_0(ix)]$
$= if_0(x)$

Hence, f_0 is linear
if $f_0(x)$ is real then
$f_0(x) = g_0(x)$
and
$|f_0(x)| = |g_0(x)|$
$\leq \sup_{\|x\| \leq 1} |g_0(x)|$
$\Rightarrow \|g_0\| \leq 1$

if $f_0(x)$ is complex then
$f_0(x) = re^{i\theta}, \forall r > 0$

$\therefore |f_0(x)| = r = e^{-i\theta} f_0(x) = f_0(e^{-i\theta}x)$.

But,
$\|e^{-i\theta}x\| = \|x\| \leq 1 \quad \forall x \in M_0$ and $\|x\| = 1$

Hence for the real valued function $f_0(e^{-i\theta}x)$ we have,

$|f_0(e^{-i\theta}x)| \leq 1$

In other words,

$|f_0(x)| \leq 1 \quad \forall\, x \in M_0$

Thus in either case,

$|f_0(x)| \leq 1 \quad$ with $\|x\| = 1$

Hence

$\|f_0\| = 1 = \|f\|$

$\therefore \boxed{\|f_0\| = \|f\|}$ Hence Proved.

* Application of Hahn-Banach Theorem

Statement :- If X is a norm space and x_0 is a ~~zero~~ non-zero vector in X then $\exists$ a continuous linear function on X such that,

$f_0(x_0) = \|x_0\|$ and $\|f_0\| = 1$

Proof :- let $M = \{x_0\}$ be a linear subspace ~~by~~ x_0 define f on M by,

$f(x) = f(\alpha x_0)$
$= \alpha \|x_0\| \quad \forall\, x \in M$.

where α is a scalar and $x \in M$. so let $x, y \in M$, then,

$f(x+y) = f(\alpha x_0 + \beta x_0)$
$= f[(\alpha+\beta) x_0]$
$= \alpha\|x_0\| + \beta\|x_0\|$
$= f(\alpha x_0) + f(\beta x_0)$
$= f(x) + f(y)$

and

$f(\lambda x) = \lambda f(x) \quad \forall\, \lambda$ scalar.

Hence, f is linear.

let $y \in M$ so that $y = \alpha x_0$ and

$\|f(y)\| = |f(\alpha x_0)|$
$= |\alpha| \|x_0\|$
$= \|y\| \leq k\|y\|, \quad k \geq 1.$

This shows that f is a bounded linear functional on M.

Also,

$\|f\| = \sup_{\|x\|=1} |f(x)|$

$= \sup_{\|\alpha x_0\|=1} |f(\alpha x_0)|$

$= \sup_{\|\alpha x_0\|=1} |\alpha| \|x_0\|$

$\Rightarrow \|f\| = 1$

Then by the Hahn banach theorem f can be extended to a continuous linear functional f_0 on X such that,

$\|f\| = \|f_0\| = 1$

moreover,

$f_0(x_0) = f(x_0) \quad \forall\, x_0 \in M$

$\Rightarrow f_0(x_0) = \|x_0\|$

This complete the proof of theorem.

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* **Theorem:**– let X be a nls and M is a closed subspace. Further assume that $w \in X - M$ ($w \in X$, but $w \notin M$) then $\exists$ $F \in X^*$ s.t.

$$F(m) = 0 \quad \forall\, m \in M$$

and

$$F(w) = 1.$$

Proof:– let $w \in X - M$ and,

$$d = \inf_{m \in M} \|w - m\|$$

since M is a closed subspace and $w \notin M$

$\therefore d > 0$.

suppose N is the subspace spanned by w and M i.e.

$n \in N$, $n = \lambda w + m$, $\lambda \in R$

Define a functional on N as,

$$f(n) = \lambda$$

To prove λ is linear and bounded

$$\begin{aligned} f(n_1 + n_2) &= f[(\lambda_1 w + m) + (\lambda_2 w + m)] \\ &= f[(\lambda_1 + \lambda_2) w + m] \\ &= (\lambda_1 + \lambda_2) \\ &= f(\lambda_1 w + m) + f(\lambda_2 w + m) \end{aligned}$$

and,

$$\begin{aligned} f(\alpha n) &= f(\alpha(\lambda w + m)) \\ &= f[\alpha \lambda w + \alpha m] \\ &= \alpha f(\lambda w + m) \\ &= \alpha f(n) \end{aligned}$$

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Thus f is linear.

In order to show that f is bounded we need to show that $\exists\, k > 0$ s.t.

$$|f(n)| \leq k\|n\|. \quad \forall\, n \in N.$$

we have,

$$\begin{aligned} \|n\| &= \|\lambda w + m\| \\ &= \|-\lambda(-w - \tfrac{m}{\lambda})\| \\ &= \|\lambda\| \, \|-\tfrac{m}{\lambda} - w\| \quad \text{—— (1)} \end{aligned}$$

$\therefore -\frac{m}{\lambda} \in M$

and $g = \inf_{m \in M} \|w - m\|$

$\therefore \|-\frac{m}{\lambda} - w\| \geq d$

from eqn (1) we have

$$\|n\| \geq |\lambda| d$$

$$\Rightarrow |\lambda| \leq \frac{\|n\|}{d}$$

$$\Rightarrow |f(n)| \leq \frac{\|n\|}{d} \qquad \{\because |f(n)| = |\lambda|\}$$

$$\Rightarrow |f(n)| \leq k\|n\| \qquad \{\because k = 1/\lambda\}$$

Hence, f is bounded

Now, if $n = 0$ then $\lambda = 1$.

Therefor, $f(w) = 1$

if $n = m \in M$ then $\lambda = 0$

Thus f is bounded linear and satisfies the condition $f(\omega) = 1$ and $\cancel{f(\omega)}$ - $f(m) = 0$.

Hence by the Hahn-Banach theorem $\exists$ $F \in X^*$ s.t. F is an extension of f and F is bounded linear

$$F(\omega) = 1 \text{ and } F(m) = 0 \quad \forall m \in M.$$

This completes the proof of theorem.

* Embedding Theorem.

Statement:- "If X is a n/s then $\exists$ a linear isometric of X into X^{**}".

Proof:- for each $x \in X$ we define,

$$F_x(f) = f(x) \quad \forall f \in X^* \quad \text{— (1)}$$

To prove that F_n is linear.

for $f, g \in X^*$ we have,

$$F_x(f+g) = (f+g)(x)$$
$$= f(x) + g(x)$$
$$= F_x(f) + F_x(g)$$

and for any scalar λ.

$$F_x(\lambda f) = \lambda f(x)$$
$$= \lambda F_x(f)$$

Hence, F_x is linear.

Also,

$$|F_x(f)| = |f(x)| \le \|f\| \, \|x\|$$

which shows that F_n is bounded.

Hence $F_x \in X^{**}$

Now, we define a natural mapping

$$\Pi : X \to X^{**}$$

by

$$\Pi(x) = F_x \quad \forall x \in X.$$

we shall show that

$$\|\Pi x\| = \|x\|.$$

if $x = 0$, this identity is trivialy ~~if x=0,~~ true, we consider the case when $x \neq 0$.

$$\|\Pi x\| = \|F_x\|$$
$$= \sup\{ |F_x(f)| : \|f\| \le 1 \}$$
$$= \sup\{ |f(x)| : \|f\| \le 1 \}$$
$$\le \sup\{ \|f\| \|x\| : \|f\| \le 1 \}$$
$$\le \|x\| \quad \text{— (2)}$$

since $x \neq 0$ then $\exists$ $f \in X^*$ s.t.

$$f(x) = \|x\| \text{ and } \|f\| = 1$$

$$\because \|x\| = f(x) \le |f(x)|$$
$$\le \sup\{ |f(x)| : \|f\| = 1 \}$$
$$= \sup\{ |F_x(f)| : \|f\| = 1 \}$$
$$= \|F_n\|.$$
$$= \|\Pi(x)\|$$

$$\Rightarrow \|x\| \le \|\Pi(x)\| \quad \text{— (3)}$$

combining eqn (2) and (3) we get.

$\|x\| = \|\Pi(x)\|$

clearly, $\|\Pi(0)\| = 0 = \|0\|$

Hence, Π preserves the norm.

To prove Π is linear :- for $x, y \in X$ we have,

$$\Pi(x+y) = F_{x+y} \longrightarrow (1)$$

$$\Pi(x+y) = F_{x+y}(f) = f(x+y)$$
$$= f(x) + f(y)$$
$$= F_x(f) + F_y(f)$$
$$= \Pi(x) + \Pi(y)$$

so by (1)

$\Pi(x+y) = F_x + F_y$

$\Rightarrow \Pi(x+y) = \Pi(x) + \Pi(y)$

and for any scalar α :

$$\Pi(\alpha x) = F_{\alpha x}$$

Then,

$$F_{\alpha x}(f) = f(\alpha x)$$
$$= \alpha f(x)$$
$$= \alpha F_x(f)$$

$\Rightarrow F_{\alpha x} = \alpha F_x$

Hence,

$$\Pi(\alpha x) = \alpha F_x$$
$$= \alpha \Pi(x)$$

Since,

$\|\Pi(x)\| = \|F_x\| \leq \|x\|$

$\therefore \Pi$ is bounded linear transformation from E into E^{**}.

To prove that Π is one-one :- Then $x, y \in X$ and $x \neq y$.

Then

$$\|\Pi(x) - \Pi(y)\| = \|\Pi(x-y)\|$$
$$= \|x-y\| \neq 0$$

$\|\Pi(x) - \Pi(y)\| \neq 0$

$\Pi(x) \neq \Pi(y)$

$\therefore \Pi$ is one-one.

Hence, the natural mapping Π is isometrically isomorphic from X into X^{**}.

* Anihilator :- let X be a norm linear space and S be a subset of X then the anihilator of S is denoted by $S^{\perp}$ is defined by

$$S^{\perp} = \{f \in X^* : f(x) = 0, \forall x \in S\}$$

unit anihilator ; denoted by S^0, is defined by

$$S^0 = \{f \in S^{\perp} : \|f\| \leq 1\}$$

we also define the sets $S_1^{\perp}$ and S_0^0 as follows

$S_1^{\perp} = \{x \in X : f(x) = 0 \ \forall f \in S^{\perp}\}$.

$S_0^0 = \{x \in S_1^{\perp} : \cancel{f(x) = 0} \ \|x\| \leq 1\}$

$S^{\perp}$ and $S_1^{\perp}$ are closed subspace of X^* and X respectively.

20) **Theorem:-** If S is a subspace of a nls X, then $\bar{S} = S^{\perp}_{\perp}$.

Proof:- clearly $S \subseteq S^{\perp}_{\perp}$ and since $S^{\perp}_{\perp}$ is closed it follows that

$$\bar{S} \subseteq S^{\perp}_{\perp} \quad \text{—(1)}$$

To show the reverse inclusion let $x \notin \bar{S}$ and define the linear functional f on the subspace generated by x and $\bar{S}$ by.

$$f(\alpha x + s) = \alpha.$$

where α is an scalar and $s \in \bar{S}$. Then norm of f on this subspace is given by.

$$\|f\| = \sup\left\{ \frac{|f(\alpha x + s)|}{\|\alpha x + s\|} : \alpha x + s \neq 0 \right\}$$

$$= \sup\left\{ \frac{|f(x+s)|}{\|x+s\|} : s \in \bar{S} \right\} \text{ for } \alpha = 1$$

$$= \frac{1}{\inf\{\|x+s\| : s \in \bar{S}\}}$$

$$= \frac{1}{\|[x]\|}$$

where $[x]$ denotes the equivalence class of x in the quotient space $X/\bar{S}$.

since, $x \notin \bar{S}$

$\therefore \|[x]\| > 0$

and hence f is bounded. Therefore f can be extended to a functional $F \in X^*$. Then we have,

$$F(x) = f(x) = 1.$$

but, $F(s) = f(s) = 0. \quad \forall s \in \bar{S}$

Hence $F \in S^{\perp}$.

But this shows that $x \notin S^{\perp}_{\perp}$ and we conclude that $S^{\perp}_{\perp} \subseteq \bar{S}$. —(2)

Then eqn (1) and (2) we have

$\boxed{S^{\perp}_{\perp} = \bar{S}}$ Hence Proved.

* **Theorem:-** A Banach space X is reflexive if and only if X^* is reflexive.

Proof:- suppose X is reflexive then we have to show that X^* is also reflexive. since X is reflexive, Therefore the cannonical mapping $\Pi : X \to X^{**}$ is onto.

Define $x^* \in X^*$ by

$$x^*(x) = x^{***}(\Pi(x)). \quad \forall x \in X.$$

for an arbitrary x^{**} we have,

$$\Pi(x) = x^{**} \quad \{\because \Pi \text{ is surjective}\}$$

Now let $\Pi^*: X^* \to X^{****}$ is the cannonical embedding of X^* then,

$$x^{***}(\Pi(x)) = x^*(x)$$
$$= \Pi(x)x^*$$
$$= (\Pi^* x^*)\Pi(x)$$

Hence $x^{***} = \Pi^*(x^*)$

consequently,

$$X^* = X^{**}$$

showing that X^* is reflexive.

Conversely: suppose that X^* is reflexive so let,

$x^{***} \in X^{***}$

let $x^* \in X^*$ be s.t.

$$\Pi^*(x^*) = x^{***}$$

This is possible because Π^* is surjective. let,

$$x^{***}(\Pi(x)) = \Pi^*(x^*)\,\Pi(x)$$
$$= x^*(x)$$
$$= 0$$

Then $x^* = 0$

Hence, $x^{***} = 0$.

So, $\Pi(x)$ is dense in X^{**} by the corollary of Hahn-Banach theorem.

$\therefore B(x)$ is closed and complete.

$\therefore \Pi(X) = X^{**}$

Hence, X is reflexive.

* **Theorem:** let X be a real n.l.s and S a is the subspace of X. Then for each $x \in X$.

$$\inf\{\|x-s\| : s \in S\} = \sup\{f(x) : f \in S^0\}$$

moreover, the supremum is attained for some $f_0 \in S^0$.

Proof: let

$$\inf\{\|x-s\| : s \in S\} = d$$

and choose a sequence $\{s_n\}$ in S such that

$$\|x - s_n\| \to d \text{ as } n \to \infty.$$

for any $f \in S^0$ we have,

$$f(x) = f(x-s_n) + f(s_n)$$
$$f(x) = f(x-s_n) \quad \{\because f(s_n) = 0\}$$
$$\le \|f\|\,\|x-s_n\|$$
$$f(x) \le \|x-s_n\| \quad \{\because f \in S^0 \text{ then } \|f\| \le 1\}$$

Hence,

$$f(x) \le d.$$

$\Rightarrow \sup\{f(x) : f \in S^0\} \le d.$

$\Rightarrow \sup\{f(x) : f \in S^0\} \le \inf\{\|x-s\| : s \in S\}$ — (1)

we complete the proof by exhibiting $f_0 \in S^0$ which satisfy:

$f_0(x) = d$

define a linear functional on the subspace generalised by x and s by

$f(\lambda x + s) = \lambda d$ where $\lambda \in R$ and $s \in S$

then,

$$\|f\| = \sup\left\{ \frac{|f(\lambda x + s)|}{\|\lambda x + s\|} : \lambda \in R, s \in S \right\}$$

$$= \sup\left\{ \frac{|\lambda d|}{\|\lambda x + s\|} : \lambda \in R, s \in S \right\}$$

$$= \sup\left\{ \frac{d}{\|x + \frac{s}{\lambda}\|} : s \in S \right\}$$

$$= \frac{d}{\inf\{ \|x + \frac{s}{\lambda}\| : s \in S \}}$$

$$= \frac{d}{d} = 1$$

$\Rightarrow \|f\| = 1$.

Therefore f may be extended to a linear functional $f_0 \in X^*$ which satisfy $\|f_0\| = 1$.

Then, $f_0(x) = f_0(1 \cdot x + 0) = d$

and for any $s \in S$, $f_0(s) = 0$

$\therefore f_0 \in S^0$

$\Rightarrow \sup\{ f_0(x) : f_0 \in S^0 \} \geq d$

$\Rightarrow \sup\{ f_0(x) : f_0 \in S^0 \} \geq \inf\{ \|x - s\| : s \in S \}$ ——— (2)

combining eqn (1) and (2) we get,

$\sup\{ f_0(x) : f_0 \in S^0 \} = \inf\{ \|x - s\| : s \in S \}$.

This complete the proof of theorem.

* **Theorem:** A closed linear subspace of a reflexive Banach space is reflexive.

Proof: Let M be a closed linear subspace of a reflexive Banach space X. Then we have to prove that M is reflexive.

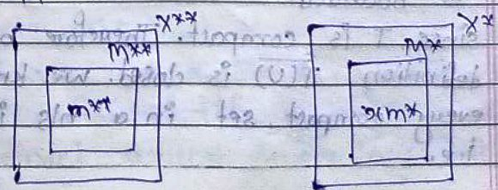
Let $T_M : M \to M^{**}$ be the canonical mapping of M into M^{**}. Then we have to show that T_M is surjective.

Let $m^{**} \in M^{**}$

Define $x^{**} \in X^{**}$ by

$x^{**}(x^*) = m^{**}(x_{m^*})$, $x^* \in X^*$

where, x_{m^*} is the restriction of x^* to M.



But $\Pi : X \to X^*$ be the surjective mapping so $\exists$ some $x \in X$ s.t.

$$\Pi(x) = x^{**}.$$

Now, we show that $x \in M$ if $x^* \in X^*$ is on M then,

$$x^*(x) = x^{**}(x^*)$$
$$= m^{**}(x m^*)$$
$$= m^{**}(0)$$
$$x^*(x) = 0$$

$\Rightarrow x \in \overline{M}$.

But then, $\Pi m(x) = M^{**}$

$\therefore \Pi m$ is surjective hence M is reflexive

★ **Theorem:**— let X and Y be nls then every compact linear operator T is a mapping from X to Y is continuous but conversely not true.

Proof:— consider the unit sphere

$$U = \{x \in X : \|x\| = 1\}$$

is bounded.

Since, T is compact. Therefore by the definition $\overline{T(U)}$ is closed. we know that every compact set in a nls is norm i.e.

$\overline{T(U)}$ is bounded

$\Rightarrow \sup\limits_{\|x\|=1} \|Tx\| \leq \infty$

$\Rightarrow T$ is bounded

$\Rightarrow T$ is continuous.

Conversely:— To prove the converse we consider the following example.

let $I : X \to X$ be the identity operator where,

$$\dim X = \infty$$

Then I is continuous.

let

$$M = \{x \in X ; \|x\| \leq 1\}$$

be a closed unit ball then M is bounded.

Now,

$$\overline{I(M)} = \overline{M} = M$$

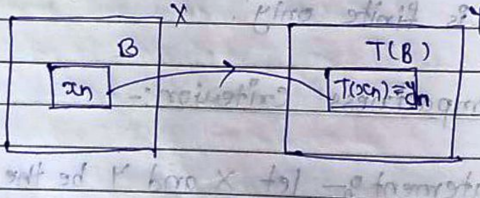
But M is not compact and since we know that "in a nls X every closed ball (with unit radius) is compact iff X is finite only".

13,15 ★ **Compactness Criterion:**—

statement:— let X and Y be the nls and $T : X \to Y$ be a linear operator then T is compact iff it maps every bounded sequence $\{x_n\}$ in X onto a sequence $\{Tx_n\}$ in Y which has a convergent subsequence.

Proof:- suppose that T is compact and $\{x_n\} \subset X$ is bounded, then by the definition of compact operator $\{Tx_n\} \subset Y$ is compact. Also by the definition of compact space $\{Tx_n\}$ contains a converges subsequence.

Conversely: Assume that every bounded sequence $\{x_n\}$ contains a subsequence $\{x_{n_k}\}$ such that; $\{Tx_{n_k}\}$ converges in Y. Then we have to show that T is compact. consider any bounded subset B in X and let $\{y_n\}$ be any sequence in $\overline{T(B)}$ in $T(B)$. Then $y_n = T(x_n)$ for some $x_n \in B$



and sequence $\{x_n\}$ is bounded. By the assumption $\{Tx_n\}$ contains ~~argument~~ convergent subsequence. Hence $\overline{T(B)}$ is compact by the definition of compact space because $\{y_n\}$ was arbitrary. Hence T is compact.

* Theorem:- (finite-dimensional domain or range)

statement:- "Let X and Y be normed space and $T: X \to Y$ be a linear operator, Then,

(a) T is bounded and $\dim T(X) < \infty$ Then the operator T is compact.

(b) If $\dim X < \infty$ then the operator T is compact.

Proof:- (a) suppose T is bounded and $\dim T(X) < \infty$ then we have to show that T is ~~complete~~ compact.

let $\{x_n\}$ be any bounded sequence in X then the inequality.

$$\|Tx_n\| \leq \|T\| \|x\|.$$

show that $\{Tx_n\}$ is bounded. Also since $\dim T(x) < \infty$.

$\therefore \{Tx_n\}$ is relatively compact.
i.e. $\overline{\{Tx_n\}}$ is compact.
Thus it follows that $\{Tx_n\}$ has a convergent subsequence.
since $\{x_n\}$ has an arbitrary bounded sequence in X.
$\because$ The operator T is compact.

Result:- In a finite dimensional norm space X, any subset $M \subset X$ is compact if and only if M is closed and bounded.

(b) we know that,
(1) A normed space X is finite dimensional then every linear operator on X is bounded.
(2) If $\dim D(T) = n < \infty$ then,
$\dim R(T) \le \dim(D(T))$
$\therefore \dim T(X) \le \dim(X) < \infty.$
~~Hence~~
Here, we have $\dim X < \infty$.
$\because$ by (1) result T is bounded.

$$\boxed{\dim T(X) \le \dim(X) < \infty}$$

Hence, by (a) T is compact.

4) Sequence of compact linear Operator:-

Statement:- let $\{T_n\}$ be a sequence of compact linear operators from a normed space X into a Banach space Y and T be a bounded linear operator, $T: X \to Y$ such that

$$\|T_n - T\| \to 0 \text{ as } n \to \infty.$$

then the limit operator T is compact.

Proof:- let $\{T_n\}$ be a sequence of compact linear operator from a norm space X into a Banach space Y s.t.

$$\|T_n - T\| \to 0 \text{ as } n \to \infty$$

In order to show that T is compact it is enough to show that for every bounded sequence $\{x_n\}$ in X the sequence $\{Tx_n\}$ has a convergent subsequence.
so let $\{x_n\}$ be a bounded sequence in X and $\varepsilon \ge 0$ be given. Then by the assumption of $\{T_n\}$ $\exists$ $n \in N$ s.t.

$$\|T_n - T\| < \varepsilon \quad \forall\, n \ge N.$$

$\because T_N$ is compact.

$\because \exists$ a subsequence $\{x_{n_k}\}$ of $\{x_n\}$ such that ~~$\{T_n T x_k$~~ $\{T_n x_{n_k}\}$ converges.

In particular $\exists$ $n_0 \in N$ s.t.

$$\|T_n x_{n_k} - T_n x_{n_l}\| < \varepsilon \quad \forall k,l$$

Now,

~~$\|T_n x_{n_k} - T_n$~~

$$\|Tx_{n_k} - Tx_{n_l}\| \le \|Tx_{n_k} - T_n x_{n_k}\| + \|T_n x_{n_k} - T_n x_{n_l}\| + \|T_n x_{n_l} - T x_{n_l}\|$$

$$\le \|T - T_n\| \, \|x_{n_k}\| + \|T_n x_{n_k} - T_n x_{n_l}\| + \|T_n - T\| \, \|x_{n_l}\|$$

$$\le \varepsilon \cdot c + \varepsilon + \varepsilon \cdot c$$
$$= (2c+1)\varepsilon$$

where $c \ge 0$ such that,
$$\|x_n\| \le c \quad \forall n \in N.$$

~~where~~ This $\{Tx_{n_k}\}$ is a cauchy sequence
$\because Y$ is complete
$\therefore \{Tx_{n_k}\}$ converges.
Thus we have to show that for bounded seq $\{x_n\}$ in X the sequence $\{Tx_n\}$ has a convergent subsequence $\{T_n x_{n_k}\}$ in Y.
$\Rightarrow T$ is compact.
This complete the proof of theorem.

* Hahn Banach theorem complex linear space:

Statement:- let E be a complex linear space and let M be a linear subspace of E. suppose q is a seminorm on E and f is a linear functional define on M s.t.

$$|f(x)| \le q(x) \quad \forall x \in M.$$

then there is a linear functional g on E such that,

$f \subset g$ and (extension)
$$\Rightarrow |g(x)| \le q(x) \quad \forall x \in E.$$

Proof:- for each $x \in M$ write
$$f(x) = f_1(x) + i f_2(x) \quad \text{--- (1)}$$
then f_1 and f_2 are real linear functional on M.
Also we have,

$$|f_1(x)| \le |f(x)| \le q(x) \quad \forall x \in M.$$

Now consider E and M as linear space and apply Hahn-Banach Theorem for real linear space obtain a real linear functional g on E s.t.
$f_1 \subset g_1$ and
$$|g_1(x)| \le q(x) \quad \forall x \in E.$$

Now define g on E by

$$g(x) = g_1(x) - i g_1(ix)$$

Clearly g is complex linear function on E. Further for $x \in N$,

$$g_1(ix) + i f_2(ix) = f_1(ix) + i f_2(ix) \quad \{\because g \text{ is extension of } f\}$$
$$= f(ix)$$
$$= i f(x)$$
$$= -f_2(x) + i f_1(x)$$

$\Rightarrow g_1(ix) + i f_2(ix) = -f_2(x) + i f_1(x)$

so that,

$g_1(ix) = -f_2(x)$ $\{\because$ comparing real part$\}$

$\because$ $g(x) = g_1(x) - i g_1(ix)$
$= f_1(x) + i f_2(x)$

$\Rightarrow g(x) = f(x)$

$\Rightarrow f \subset g$.

it remains only to show that,

$|g(x)| \le q(x)$ $\forall$ $x \in E$

let $x \in E$ be arbitrary write,

$g(x) = re^{i\theta}$, $r \ge 0$, $\theta \in \mathbb{R}$

Then,

$|g(x)| = r = e^{-i\theta} g(x)$

$= g(e^{-i\theta} x)$

$= g_1(e^{-i\theta} x)$ $\{\because$ by def. of $g\}$

$\le q(e^{-i\theta} x)$

$= |e^{-i\theta}|\, q(x)$

$= q(x)$

$\Rightarrow |g(x)| \le q(x)$.

2011 * ~~Theorem:- let S be a subspace of n/s X then prove that $\bar{S} = S^{\perp}_{\perp}$~~

* Theorem:- Any finite dimensional space is reflexive.

Proof:- let X be a n dimensional n/s and suppose $\beta = \{e_1, e_2, \dots, e_n\}$ is a basis for X.

let $B^* = \{f_1, f_2, \dots, f_n\}$ consisting of this functional defined by

$$f_j(x) = x_j$$

where $x = \{x_1e_1, x_2e_2, \dots, x_ne_n\}$

or, f_j is defined by

$$f_j(e_i) = \delta_{ij} = \begin{cases} 1 & \text{if } i = j \\ 0 & \text{if } i \ne j \end{cases}$$

then it is easy to show that B^* is the basis for

i.e. X is a n/s of dimensional n then dimension of its dual space is also n.

so, dimention of X^{**} is also n.

Now, X is isometrically isomorphic to

$\Pi(X) \subset X^{**}$

$\because$ $\dim \Pi(X)$ is n

Hence $\Pi(X) = X^{**}$.

* Theorem:- let X be a reflexive Banach space then X is separable $\Leftrightarrow$ X^* is separable.

Proof:- given that X be a reflexive Banach space.

if X is seperable then it is reflexive

$X^{**}(X)$ is separable. But we know that if X^* is separable then X is separable.

Conversely:- Assume that X is seperable.

since X is reflexive.

X^{**} is isometric to X.

Hence $X^{**} = (X^*)^*$ is separable.

Thus X^* is separable.

* Theorem:- If X is reflexive Banach space then any $f \in X^*$ attains its norm on the unit Ball.

Proof:- let X be reflexive Banach space and let $f \in X^*$

case I:- if $f = 0$

then its norm zero is attains each vector on the unit ball.

case II:- Now, suppose that $f \neq 0$.

then by Hahn-Banach theorem $F \in X^{**}$.

s.t. $\|F\| = 1$

$F(f) = \|f\|$.

$\because X$ is reflexive, $\exists\, x \in X$ s.t.

$F_x = F$ then

$\|x\| = \|\Pi(x)\|$

$= \|F_x\| = \|F\| = 1$.

$\therefore f(x) = F_x(f) = F(f) = \|f\|$.